

A DESCRIPTION OF n -ARY SEMIGROUPS POLYNOMIAL-DERIVED FROM INTEGRAL DOMAINS

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ABSTRACT. We provide a complete classification of the n -ary semigroup structures defined by polynomial functions over infinite commutative integral domains with identity, thus generalizing Głazek and Gleichgewicht's classification of the corresponding ternary semigroups.

1. INTRODUCTION

Let R be an infinite commutative integral domain with identity and let $n \geq 2$ be an integer. In this note we provide a complete description of all the n -ary semigroup structures defined by polynomial functions over R (i.e., the n -ary semigroup structures polynomial-derived from R).

For any integer $k \geq 1$, let $[k] = \{1, \dots, k\}$. Recall that a function $f: R^n \rightarrow R$ is said to be *associative* if it solves the following system of $n-1$ functional equations:

$$(1) \quad \begin{aligned} & f(x_1, \dots, f(x_i, \dots, x_{i+n-1}), \dots, x_{2n-1}) \\ &= f(x_1, \dots, f(x_{i+1}, \dots, x_{i+n}), \dots, x_{2n-1}), \quad i \in [n-1]. \end{aligned}$$

In this case, the pair (R, f) is called an *n -ary semigroup*.

The introduction of n -ary semigroups goes back to Dörnte [1] and led to the generalization of groups to n -ary groups (polyadic groups). The next extensive study on polyadic groups was due to Post [10] and was followed by several contributions towards the description of n -ary groups and similar structures. To mention a few, see [2–6, 8, 9, 11].

We now state our main result, which provides a description of the possible associative polynomial functions from R^n to R . Let $\text{Frac}(R)$ denote the fraction field of R and let $\mathbf{x} = (x_1, \dots, x_n)$.

Main Theorem. *A polynomial function $p: R^n \rightarrow R$ is associative if and only if it is one of the following functions:*

- (i) $p(\mathbf{x}) = c$, where $c \in R$,
- (ii) $p(\mathbf{x}) = x_1$,
- (iii) $p(\mathbf{x}) = x_n$,
- (iv) $p(\mathbf{x}) = c + \sum_{i=1}^n x_i$, where $c \in R$,
- (v) $p(\mathbf{x}) = \sum_{i=1}^n \omega^{i-1} x_i$ (if $n \geq 3$), where $\omega \in R \setminus \{1\}$ satisfies $\omega^{n-1} = 1$,
- (vi) $p(\mathbf{x}) = -b + a \prod_{i=1}^n (x_i + b)$, where $a \in R \setminus \{0\}$ and $b \in \text{Frac}(R)$ satisfy $ab^n - b \in R$ and $ab^k \in R$ for every $k \in [n-1]$.

Date: January 27, 2011.

2010 Mathematics Subject Classification. 20N15.

Key words and phrases. n -ary semigroup, n -ary group, associativity, polynomial function.

Communicated by Mikhail Volkov.

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The Main Theorem shows that the associative polynomial functions of degree greater than 1 are symmetric, i.e., invariant under any permutation of the variables.

Example 1. The third-degree polynomial function $p: \mathbb{Z}^3 \rightarrow \mathbb{Z}$ defined on the ring \mathbb{Z} of integers by

$$p(x_1, x_2, x_3) = 9x_1x_2x_3 + 3(x_1x_2 + x_2x_3 + x_3x_1) + x_1 + x_2 + x_3$$

is associative since it is the restriction to \mathbb{Z} of the associative polynomial function $q: \mathbb{Q}^3 \rightarrow \mathbb{Q}$ defined on the field \mathbb{Q} of rationals by

$$q(x_1, x_2, x_3) = -\frac{1}{3} + 9 \prod_{i=1}^3 \left(x_i + \frac{1}{3}\right).$$

The classification given in the Main Theorem was already obtained for ternary semigroups (i.e., when $n = 3$) by Głazek and Gleichgewicht [7]. Surprisingly, the classification for arbitrary n remains essentially the same except for certain solutions of type (v) (already mentioned in [1]), whose existence is subordinate to that of nontrivial roots of unity. Note that, when n is odd, (v) always provides the solution

$$p(\mathbf{x}) = \sum_{i=1}^n (-1)^{i-1} x_i,$$

which was the unique solution of type (v) found in [7] for $n = 3$.

In Section 2 we give the proof of the Main Theorem. In Section 3 we analyze some properties of these n -ary semigroup structures: we show that they are medial, determine the n -ary groups defined by polynomial functions, and discuss irreducibility issues for these n -ary semigroups.

2. TECHNICALITIES AND PROOF OF THE MAIN THEOREM

Throughout this section, with every function $f: R^n \rightarrow R$ we associate n functions $f_i: R^{2n-1} \rightarrow R$, $i \in [n]$, defined by

$$(2) \quad f_i(x_1, \dots, x_{2n-1}) = f(x_1, \dots, f(x_i, \dots, x_{i+n-1}), \dots, x_{2n-1}).$$

It follows that f is associative if and only if $f_1 = \dots = f_n$.

It is clear that the definition of R enables us to identify the ring $R[x_1, \dots, x_n]$ of polynomials of n indeterminates over R with the ring of polynomial functions of n variables from R^n to R . Recall that, for any integer $d \geq 0$, a polynomial function $p: R^n \rightarrow R$ of degree $\leq d$ can be written as

$$p(\mathbf{x}) = \sum_{j_1 + \dots + j_n \leq d} c_{j_1, \dots, j_n} x_1^{j_1} \dots x_n^{j_n}, \quad c_{j_1, \dots, j_n} \in R.$$

For every $i \in [n]$, we denote the degree of p in x_i by $\deg(p, x_i)$. We also denote the degree of p by $\deg(p)$.

Proposition 2. *For every associative polynomial function $p: R^n \rightarrow R$, we have $\deg(p, x_i) \leq 1$ for every $i \in [n]$. Moreover, if $\deg(p, x_n) = 0$ (resp. $\deg(p, x_1) = 0$), then p is either a constant or the projection on the first (resp. the last) coordinate.*

Proof. Let $p: R^n \rightarrow R$ be an associative polynomial function and let p_1, \dots, p_n be the polynomial functions associated with p as defined in (2). For every $i \in [n]$, we let $d_i = \deg(p, x_i)$. By associativity, we have

$$\begin{aligned} d_i &= \deg(p_i, x_1) = \deg(p_1, x_1) = d_1^2, & i \in [n] \setminus \{1\}, \\ d_n &= \deg(p_i, x_{2n-1}) = \deg(p_n, x_{2n-1}) = d_n^2, & i \in [n] \setminus \{n\}, \end{aligned}$$

which shows that $d_1 \leq 1$ and $d_n \leq 1$.

Again by associativity, we have

$$\begin{aligned} d_i d_{n-i+1} &= \deg(p_i, x_n) = \deg(p_1, x_n) = d_1 d_n, & i \in [n], \\ d_i &= \deg(p_{i+1}, x_i) = \deg(p_i, x_i) = d_1 d_i, & i \in [n-1], \\ d_i &= \deg(p_{i-1}, x_{n+i-1}) = \deg(p_i, x_{n+i-1}) = d_i d_n, & i \in [n] \setminus \{1\}. \end{aligned}$$

If $d_1 = d_n = 1$, then the first set of equations shows that $d_i = 1$ for every $i \in [n]$. If $d_n = 0$, then the third set of equations shows that p is of the form $p(\mathbf{x}) = c_1 x_1 + c_0$ and hence we can conclude immediately. We proceed similarly if $d_1 = 0$. \square

By Proposition 2 an associative polynomial function $p: R^n \rightarrow R$ can always be written in the form

$$p(\mathbf{x}) = \sum_{j_1, \dots, j_n \in \{0,1\}} c_{j_1, \dots, j_n} x_1^{j_1} \cdots x_n^{j_n}, \quad c_{j_1, \dots, j_n} \in R.$$

Using subsets of $[n]$ instead of Boolean indexes, we obtain

$$(3) \quad p(\mathbf{x}) = \sum_{J \subseteq [n]} c_J \prod_{j \in J} x_j, \quad c_J \in R.$$

In order to prove the Main Theorem, we only need to describe the class of associative polynomial functions of the form (3).

To avoid cumbersome notation, for every subset $S = \{j_1, \dots, j_k\}$ of integers and every integer m , we set $S + m = \{j_1 + m, \dots, j_k + m\}$. Also, for every $i \in [n]$, we let

$$\begin{aligned} A_i &= \{1, \dots, i-1\} = [i-1], \\ B_i &= \{i, \dots, i+n-1\} = [n] + i - 1, \\ C_i &= \{i+n, \dots, 2n-1\} = [n-i] + n + i - 1, \end{aligned}$$

with the convention that $A_1 = C_n = \emptyset$.

Lemma 3. *If $p: R^n \rightarrow R$ is of the form (3), then for every $i \in [n]$ the associated function $p_i: R^{2n-1} \rightarrow R$ is of the form*

$$(4) \quad p_i(x_1, \dots, x_{2n-1}) = \sum_{S \subseteq [2n-1]} r_S^i \prod_{j \in S} x_j$$

and its coefficients are given in terms of those of p by

$$r_S^i = \begin{cases} c_{J_S^i \cup \{i\}} c_{K_S^i}, & \text{if } S \cap B_i \neq \emptyset, \\ c_{J_S^i \cup \{i\}} c_\emptyset + c_{J_S^i}, & \text{otherwise,} \end{cases}$$

where $J_S^i = (S \cap A_i) \cup ((S \cap C_i) - n + 1)$ and $K_S^i = (S \cap B_i) - i + 1$.

Proof. We first note that

$$p(x_i, \dots, x_{n+i-1}) = \sum_{K \subseteq [n]} c_K \prod_{k \in K} x_{k+i-1} = \sum_{K \subseteq [n]} c_K \prod_{k \in K+i-1} x_k.$$

Then, partitioning $J \subseteq [n]$ into $J \cap A_i$, $J \cap \{i\}$, and $J \cap (C_i - n + 1)$, we obtain

$$\begin{aligned} p_i(x_1, \dots, x_{2n-1}) &= \sum_{J \subseteq [n]} c_J \prod_{j \in J \cap A_i} x_j \prod_{j \in (J \cap \{i\}) \cap C_i} x_j \prod_{j \in J \cap \{i\}} \left(\sum_{K \subseteq [n]} c_K \prod_{k \in K+i-1} x_k \right) \\ &= \sum_{J \subseteq [n], J \ni i} \sum_{K \subseteq [n]} c_J c_K \prod_{j \in J \cap A_i} x_j \prod_{j \in (J \cap \{i\}) \cap C_i} x_j \prod_{k \in K+i-1} x_k \\ &\quad + \sum_{J \subseteq [n], J \not\ni i} c_J \prod_{j \in J \cap A_i} x_j \prod_{j \in (J \cap \{i\}) \cap C_i} x_j. \end{aligned}$$

The result is then obtained by reading the coefficient of $\prod_{i \in S} x_i$ in the latter expression. \square

Proposition 4. *Let $p: R^n \rightarrow R$ be an associative polynomial function of the form (3). If $c_{[n]} = 0$, then $\deg(p) \leq 1$.*

Proof. We assume that $c_{[n]} = 0$ and prove by induction that $c_J = 0$ for every $J \subseteq [n]$ such that $|J| \geq 2$. Suppose that $c_J = 0$ for every $J \subseteq [n]$ such that $|J| \geq k$ for some $k \geq 3$. Fix $J_0 \subseteq [n]$ such that $|J_0| = k - 1$. We only need to show that $c_{J_0} = 0$.

Assume first that $\ell = \min(J_0) \leq (n + 1)/2$.

(i) Case $\ell = 1$. Let

$$S = J_0 \cup ((J_0 + n - 1) \setminus \{n\}) \subseteq [2n - 1].$$

We have $S \cap A_1 = \emptyset$, $S \cap B_1 = J_0$, and $(S \cap C_1) - n + 1 = J_0 \setminus \{1\}$. By Lemma 3, we have $r_S^1 = c_{J_0}^2$. Setting $m = \min([n] \setminus J_0)$,¹ we also have $|S \cap A_m| = |A_m| = m - 1$ and $|(S \cap C_m) - n + 1| = |J_0| - (m - 1)$. Moreover, $S \cap B_m \neq \emptyset$ for otherwise we would have $J_0 = \{1\}$, which contradicts $|J_0| \geq 2$. Thus, using Lemma 3, associativity, and the induction hypothesis, we have $r_S^m = 0$ and therefore

$$c_{J_0}^2 = r_S^1 = r_S^m = 0.$$

(ii) Case $1 < \ell \leq (n + 1)/2$. Let

$$S = (J_0 + \ell - 1) \cup ((J_0 + n - 1) \setminus \{n + \ell - 1\}) \subseteq [2n - 1].$$

We proceed as above to obtain $r_S^\ell = c_{J_0}^2$. By associativity it is then sufficient to show that $r_S^{2\ell-1} = 0$. Using the notation of Lemma 3, we can readily see that $|K_S^{2\ell-1}| \geq |J_0|$. Hence by Lemma 3 we only need to show that $c_{K_S^{2\ell-1}} = 0$.

If $|K_S^{2\ell-1}| > |J_0|$, then we conclude by using the induction hypothesis. If $|K_S^{2\ell-1}| = |J_0|$, then we can apply case (i) since $\min(K_S^{2\ell-1}) = 1$.

If $\ell > (n + 1)/2$, we proceed symmetrically by setting $\ell' = \max(J)$ and considering the cases $\ell' = n$ and $(n + 1)/2 \leq \ell' < n$ separately. \square

Proposition 5. *A polynomial function $p: R^n \rightarrow R$ of the form (3) with $c_{[n]} = 0$ is associative if and only if it is one of the functions of types (i)–(v).*

Proof. It is straightforward to see that the functions of types (i)–(v) are associative polynomial functions.

Now, by Proposition 4 the polynomial function p has the form

$$p(\mathbf{x}) = c_0 + \sum_{i=1}^n c_i x_i, \quad c_0, \dots, c_n \in R.$$

Comparing the coefficients of x_1 in p_1 and p_2 , we obtain the equation $c_1^2 = c_1$. Similarly, we show that $c_n^2 = c_n$. If $c_1 = 0$ or $c_n = 0$, we conclude by Proposition 2. Thus we can assume that $c_1 = c_n = 1$. Comparing the coefficients of x_i in p_i and p_{i-1} for $2 \leq i \leq n$, we obtain the equations $c_1 c_i = c_2 c_{i-1}$, or equivalently, $c_i = c_2^{i-1}$ and $c_2^{n-1} = 1$. Finally, since the constant term in p_i is $c_0 + c_i c_0$, we must have $c_0 = 0$ unless $c_1 = \dots = c_n$. \square

¹In fact, $m = \text{mex}_{[n]}(J_0)$, where ‘mex’ stands for the *minimal excluded number*, well known in combinatorial game theory.

Lemma 6. *Let $p: R^n \rightarrow R$ be an associative polynomial function of the form (3). If $c_{[n]} \neq 0$, then p is a symmetric function.*

Proof. Let us first prove that $c_J = c_{J'}$ for every $J, J' \in [n]$ such that $|J| = |J'| = n-1$. Setting $S = [2n-1] \setminus \{n\}$, we see by Lemma 3 that $r_S^i = c_{[n]}c_{[n] \setminus \{n-i+1\}}$ for $i \in [n]$ and we conclude by associativity.

We now proceed by induction. Suppose that $c_J = c_{J'}$ for every $J, J' \in [n]$ such that $|J| = |J'| \geq k$ for some $2 \leq k \leq n-1$ and set $c_{|J|} = c_J$ for every $J \subseteq [n]$ such that $|J| \geq k$. Fix J_0 such that $|J_0| = k-1$ and set $S = J_0 \cup C_1$ and $m = \min([n] \setminus J_0) \leq n-1$. By Lemma 3 and associativity we have $c_{[n]}c_{J_0} = r_S^1 = r_S^{m+1} = c_{n-1}c_{|J_0|+1}$. \square

The interest of Lemma 6 is shown by the following obvious result.

Lemma 7. *A symmetric function $f: R^n \rightarrow R$ is associative if and only if the associated functions f_1, \dots, f_n satisfy the condition $f_1 = f_2$.*

Recall that the n -variable elementary symmetric polynomial functions of degree $k \leq n$ are defined by

$$P_k(\mathbf{x}) = \sum_{K \subseteq [n], |K|=k} \prod_{i \in K} x_i.$$

Proposition 8. *A polynomial function $p: R^n \rightarrow R$ such that $\deg(p) > 1$ is associative if and only if it is of the form*

$$(5) \quad p(\mathbf{x}) = \sum_{k=0}^n c_k P_k(\mathbf{x}),$$

where the coefficients $c_k \in R$ satisfy the conditions

$$(6) \quad c_{j+1}c_k + c_j\delta_{k,0} = c_jc_{k+1}, \quad j \in [n-1], \quad k \in [n]-1.$$

Proof. By Proposition 5 and Lemma 6, any associative polynomial function $p: R^n \rightarrow R$ such that $\deg(p) > 1$ is of the form (5). By Lemma 7, such a polynomial function is associative if and only if $p_1 = p_2$, that is, with the notation of Lemma 3, $r_S^1 = r_S^2$ for every $S \subseteq [2n-1]$.

Set $j = |S \cap C_1|$, $k = |S \cap B_1|$, $j' = |S \cap A_2| + |S \cap C_2|$, and $k' = |S \cap B_2|$. We have either $j' = j-1$ and $k' = k+1$, or $j' = j+1$ and $k' = k-1$, or $j' = j$ and $k' = k$. Therefore we get the equations

$$\begin{aligned} c_{j+1}c_k + c_j\delta_{k,0} &= c_jc_{k+1}, & j \in [n-1], \quad k \in [n]-1, \\ c_{j+2}c_{k-1} + c_{j+1}\delta_{k-1,0} &= c_{j+1}c_k, & j \in [n-1]-1, \quad k \in [n]. \end{aligned}$$

We conclude by observing that both sets of conditions are equivalent. \square

Let us now consider the special case where R is a field.

Proposition 9. *Assume that R is a field. The associative polynomial functions from R^n to R of degree > 1 are of the form*

$$(7) \quad p_{a,b}(\mathbf{x}) = -b + a \prod_{i=1}^n (x_i + b),$$

where $a \in R \setminus \{0\}$ and $b \in R$.

Remark 1. The functions $p_{a,b}$ defined in (7) can be written in several equivalent forms. It is easy to see that they are associative since so are $p_{a,0}$ and $p_{a,b} = \varphi \circ p_{a,0} \circ (\varphi^{-1}, \dots, \varphi^{-1})$ where $\varphi(x) = x - b$.

Proof. Since the coefficient c_n in (5) is nonzero by Proposition 5, we can set $a = c_n$ and $b = c_{n-1}/a$. Using equation (6) for $j = n - 1$ and $k \geq 1$, we obtain $c_k = b c_{k+1}$, that is, $c_k = ab^{n-k}$. Using again equation (6) for $j = n - 1$ and $k = 0$, we obtain $c_0 = -b + ab^n$. We conclude by observing that the function $p_{a,b}$ is associative (see remark above). \square

We see from the proof of Proposition 9 that the system of equations (6) has a unique solution in $\text{Frac}(R)$. Therefore we can characterize the associative polynomial functions of degree > 1 as the restrictions to R of nonzero multiples of the product function, up to an affine transformation in $\text{Frac}(R)$.

Proposition 10. *Any associative polynomial function $p: R^n \rightarrow R$ such that $\deg(p) > 1$ is of type (vi).*

3. FURTHER PROPERTIES

We now investigate a few properties of the semigroup structures that we have determined.

3.1. n -ary groups. After classifying the ternary semigroups defined by polynomial functions, Glazek and Gleichgewicht [7] determined the corresponding ternary groups. Using the Main Theorem, we can also derive a description of the n -ary groups defined by polynomial functions. Recall that an n -ary quasigroup is given by a nonempty set G and an n -ary operation $f: G^n \rightarrow G$ such that for every $a_1, \dots, a_n, b \in G$ and every $i \in [n]$ the equation

$$(8) \quad f(a_1, \dots, a_{i-1}, z, a_{i+1}, \dots, a_n) = b,$$

has a unique solution in G . An n -ary group is then an n -ary quasigroup (G, f) that is also an n -ary semigroup. Recall also that in an n -ary group, with any element x is associated the element \bar{x} , called *skew to x* , defined by the equation $f(x, \dots, x, \bar{x}) = x$.

Proposition 11. *The n -ary groups (R, p) defined by polynomial functions $p: R^n \rightarrow R$ of degree ≤ 1 are of type (iv) with $\bar{x} = (2 - n)x - c$ and type (v) with $\bar{x} = x$.*

Proof. We immediately see that the polynomials of types (i)–(iii) do not define n -ary groups. It is well known that the polynomials of types (iv) and (v) define n -ary groups.² \square

In general, the n -ary semigroups (R, p) defined by type (vi) are not n -ary groups. In the special case where R is a field, we have the following immediate result.

Proposition 12. *If R is a field, the n -ary semigroup $(R \setminus \{-b\}, p_{a,b})$, where $p_{a,b}$ is defined in (7), is an n -ary group. It is isomorphic to $(R \setminus \{0\}, p_{a,0})$.*

3.2. Medial n -ary semigroup structures. We observe that all the n -ary semigroup structures given in the Main Theorem are medial. This is a general fact for functions of degree ≤ 1 on a commutative ring. This is also immediate for the function $p_{a,b}$ defined in (7) because it is the restriction to R of an n -ary operation that is isomorphic to a nonzero multiple of the n -ary product operation on $\text{Frac}(R)$. From this observation it follows that, for the n -ary groups given in Proposition 11, the map $x \mapsto \bar{x}$ is an endomorphism.

²Polynomial functions of type (v) were already considered by Dörnte [1, p. 5] in the special case of complex numbers.

3.3. (Ir)reducibility of n -ary semigroup structures. Recall that if (G, \circ) is a semigroup, then there is an obvious way to define an n -ary semigroup by $f(x_1, \dots, x_n) = x_1 \circ \dots \circ x_n$. In this case, the n -ary semigroup (G, f) is said to be *derived* from (G, \circ) or *reducible* to (G, \circ) , otherwise it is said to be *irreducible*. It is clear that the n -ary semigroups defined in types (i)–(iii) are derived from the corresponding semigroups. However, the n -ary semigroups defined in type (v) are not reducible. Indeed, otherwise we would have $y = y \circ 0 \circ \dots \circ 0$ for all $y \in R$, and therefore

$$x \circ y = x \circ (y \circ 0 \circ \dots \circ 0) = x \circ (y \circ 0) \circ 0 \circ \dots \circ 0 = x + \omega(y \circ 0),$$

for $x, y \in R$, which leads to $x \circ y = x + \omega y + c$, where $c = \omega^2(0 \circ 0)$. We know from the Main Theorem that this function does not define a semigroup. We can prove similarly that the n -ary semigroups defined in type (iv) are reducible if and only if $c = (n-1)c_0$ for $c_0 \in R$ and, when R is a field, that the semigroup $(R \setminus \{0\}, p_{a,0})$ is derived from a semigroup if and only if $a = a_0^{n-1}$ for $a_0 \in R$.

ACKNOWLEDGMENTS

This research is supported by the internal research project F1R-MTH-PUL-09MRDO of the University of Luxembourg.

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